Copy No. 79

INSTITUTE FOR COMPUTER APPLICATIONS IN

SCIENCE AND ENGINEERING (ICASE)

7N-48-CR

210979

38P

FINITE ELEMENT APPROXIMATION OF OCEAN CIRCULATION PROBLEMS

By

George J. Fix

(NASA-CR-179998) FINITE ELEMENT APPROXIMATION OF OCEAN CIRCULATION PROBLEMS (ICASE) 38 p

N89-70861

Unclas

00/48 0210979

December 11, 1973

FINITE ELEMENT APPROXIMATION OF OCEAN CIRCULATION PROBLEMS

Prepared By

George J. Fix

APPROVED BY: James M. Ortega, Director
Institute for Computer Applications
in Science and Engineering

This paper was prepared as a result of work performed under NASA Grant NGR 47-102-001 while the author was in residence at ICASE, NASA Langley Research Center.

FINITE ELEMENT APPROXIMATION OF OCEAN CIRCULATION PROBLEMS

By

George J. Fix

ICASE, Langley Research Center and

University of Michigan

1. Introduction - The study of large scale ocean flows has traditionally been a subject where perturbation techniques have played a central role [1]. The reason is that such flows are rich in boundary layers ([2],[3]), the Gulf Stream being the most striking example. While a great deal has been leanned with these analytical methods, still unresolved and as a consequence of central importance in geophysical fluid mechanics, is the dynamics of ocean flows. This is especially true for the medium or meso-scales.

Thus for example a rather eloquent theory has been developed with singular perturbation expansions which explains why there is a westward intensification of ocean currents in the Atlantic; that is, a Gulf Stream [2]. Undetected by this theory, however, is the so-called meandering of the Gulf Stream. The latter is a meso-scale phenomena, highly nonlinear, and time dependent, which is felt to have important consequences on the global circulation [4].

Another example of a meso-scale flow whose dynamics is not understood is the mid-ocean eddies. These are felt to be quite important since they apparently transport a significant amount of energy. Unfortunately they are essentially nonlinear in character and as a consequence have eluded exact analysis.

In the last few years there has been increasing interest in using numerical methods to study meso-scale dynamics [5]. There are of course many "world ocean" or global circulation models in existence (*), and while these have made a significant contribution to oceanography they can not be expected to resolve meso-scale effects on the present generation of computers. As a consequence, attention is being given to limited region models. This activity has received a fundamental stimulus from the M.O.D.E. project (mid-ocean dynamics experiment) [6], and in fact, these experiments were primarily designed to be a data base for numerical models.

In this paper we describe one such model based on the finite element method. The latter can at best be said to be untested as regards fluid calculations, although there have been some preliminary studies ([8], [9], [10], [11]). We do not wish to confront in this paper the rather controversial question of finite elements vs. finite differences in the context of computational fluid dynamics. Indeed, it may very well turn out that finite elements will play only a very small role in the latter.

What finite elements does bring to flow problems, on the other hand, is a systematic procedure for developing stable higher order approximations

(*)See the articles by K. Bryan and M. Cox in [5].

even in the presence of irregular grids and geometries. The latter are of crucial importance in the limited region model. Indeed, one wants the flexibility in choosing boundaries (to coincide for example with a streamline in the M.O.D.E. data), and irregular grids are essential since meso-scale eddies will typically occupy a small part of the M.O.D.E. region. Moreover, the inflow and outflow regions change quite substantially in time, and as we shall show in the sequel, finite elements treat this in a natural and stable manner.

It has been argued [12] that the stability and reliability of the finite element method comes at the price of efficiency. For a given problem there are always difference schemes with equivalent accuracy but requiring less work than finite elements. We find this argument rather convincing for regular grids and geometries, but much less so for the setting described above. In addition, for problems such as the one described in this paper where scientific understanding of physical phenomena is the primary goal, a premium is naturally placed on reliability.

In this paper we shall discuss only a two dimensional barotropic model [14], although the calculations with the M.O.D.E. data will be done with a three dimensional quasi-geostrophic model [13]. The reason for this is that the barotropic model can be defined simply and concisely and yet contains all the significant nonlinearities of the three dimensional model. In fact, the latter has a stratified vertical structure, and is in essence a system of barotropic models.

2. <u>The limited region barotropic model</u>. Let Ω be a region in the ocean having the boundary Γ (see Figure 2.1). In keeping with standard notation we let $y > 0$ be the northward direction and $x > 0$ be eastward. The physical variables of interest are the x and y

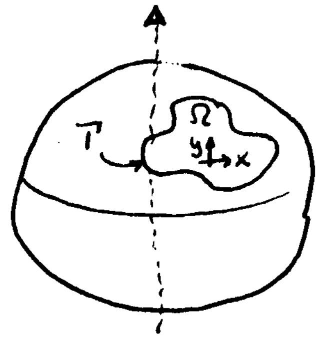

Figure 2.1.- The region Ω.

components u, v of the velocity, the pressure p, and the density ρ. Letting $f = \alpha + \beta y$ denote the Coriolis parameter the classical barotropic equations of motion are

$$(2.1)\qquad (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$(2.2)\qquad (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})v + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$(2.3)\qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0.$$

In the study of meso-scale eddies associated with oceanic flow the most important physical quantity is the potential vorticity ζ, which in our bartropic model is defined as

$$(2.4)\qquad \zeta = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) + f.$$

The first term in the brackets is the classical vorticity, and the second term represents the contribution from the Earth's rotation.

The equation describing the transport of vorticity is obtained by taking the curl of the momentum equations (2.1)-(2.2). This gives

$$(2.5) \qquad \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right)\zeta = 0.$$

Because the flow is incompressible, there is a stream function ψ satisfying

$$(2.6) \qquad u = -\frac{\partial \psi}{\partial y}, \; v = \frac{\partial \psi}{\partial x}.$$

It follows from (2.4) and (2.6) that the stream function and the potential vorticity are related through the elliptic equation

$$(2.7) \qquad \Delta\psi = \zeta - f.$$

We shall use (2.5) and (2.7) as our basic equations of motion.

The region Ω is not the world ocean but rather a smaller subset determined by the location of the M.O.D.E. experiments. As a consequence Γ is not a physical boundary in the usual sense but rather a curve in a larger oceanic body. This means that our boundary condition will be slightly different from the usual ones that occur in fluid dynamics.

In particular, we observe that at each time $t > 0$ we must solve (2.7) for the stream function ψ in terms of the potential vorticity ζ. We can do this once the flow normal to Γ is specified. Indeed, it follows from (2.6) that the velocity normal to Γ is precisely the tangential derivative of the stream function, denoted $\frac{\partial \psi}{\partial \sigma}$ in the sequel (see Figure 2.2). Hence the specification

$$\psi(\underline{x}, t) = \psi_\Gamma(\underline{x}, t) \quad \text{for } \underline{x} = (x, y) \in \Gamma,\ t > 0 \tag{2.8}$$

uniquely determines the velocity normal to Γ, and at the same time allows us to solve (2.7) for ψ in terms of ζ.

The condition (2.8) by itself is not sufficient to uniquely determine the velocity field. To see this let us note that (2.5) can be written

$$\frac{D}{Dt}(\zeta) = 0, \tag{2.5'}$$

where $\frac{D}{Dt}$ is the material time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y};$$

i.e., the derivative with respect to time following a fixed particle in space. The relation (2.5') states that the potential vorticity ζ is conserved along particle path; i.e., once a fluid particle enters Ω its potential vorticity is fixed and remains the same until it leaves Ω. This suggests that the right condition on the vorticity is to specify its values on the inflow. The latter is defined as those points on Γ where the normal velocity is negative, or what is the same

$$\Gamma_{in}(t) = \{\underline{x} = (x, y) \in \Gamma \mid \frac{\partial \psi_\Gamma}{\partial \sigma}(\underline{x}, t) \le 0\}. \tag{2.9}$$

Our condition is thus

(2.10) $$\zeta(\underline{x}, t) = \zeta_\Gamma(\underline{x}, t) \text{ for } \underline{x} \in \Gamma_{in}(t),\ t > 0 .$$

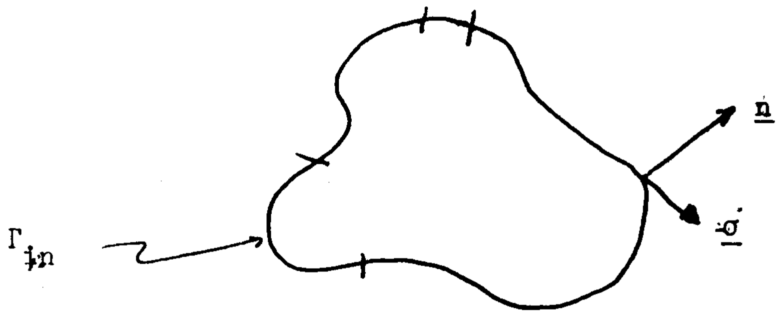

Figure 2.2.- Inflow region.

A more fundamental calculation justifying the boundary condition (2.10) starts with

(2.11) $$||\zeta(\cdot, t)||_o^2 = \iint_\Omega |\zeta(\underline{x}, t)|^2 \, d\underline{x} ,$$

which in the geophysical fluid dynamics literature is often called the potential entrosphy. Mathematically, it is of course nothing more than the spatial L_2 norm of the potential vorticity at each time $t \geq 0$. In the sequel we shall retain the notation implied in (2.11) with

$$||g||_o = \left\{ \iint_\Omega |g(\underline{x})|^2 \, d\underline{x} \right\}^{1/2}$$

denoting the L_2 norm of g, and more generally

$$||g||_r = \left\{ \sum_{r_1+r_2\leq r} \left|\left| \frac{\partial^{r_1+r_2} g}{\partial x^{r_1} \partial y^{r_2}} \right|\right|_o^2 \right\}^{1/2}$$

denoting the Sobolev norm of integral order $r \geq 0$.

To bound the entrosphy we rewrite (2.5) as

$$\frac{\partial \zeta}{\partial t} = J(\zeta, \psi), \tag{2.12}$$

where

$$J(v, w) = \frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}. \tag{2.13}$$

Multiplying (2.12) by ζ and integrating over Ω gives

$$\frac{1}{2}\frac{\partial}{\partial t} \; ||\zeta(\cdot, t)||_o^2 = \iint_\Omega J(\zeta, \psi)\zeta \; ; \tag{2.14}$$

i.e., the time growth of the potential entrosphy is governed by the values of the volume integral

$$\iint_\Omega J(\zeta, \psi)\zeta \; .$$

The key step in this calculation is to simplify this expression by converting it into a boundary integral over Γ. We observe that

$$J(\zeta, \psi)\zeta = \frac{1}{2}\frac{\partial}{\partial x}[\zeta]^2 \frac{\partial \psi}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}[\zeta]^2 \frac{\partial \psi}{\partial x}$$

Hence an integration by parts gives

$$\frac{\partial}{\partial t}\left\{||\zeta(\cdot, t)||_o^2\right\} = -\oint_\Gamma \left(\frac{\partial \psi}{\partial \sigma}\right) \zeta^2 \, , \tag{2.15}$$

which is the concise form of the law governing the time growth of entrosphy. There are a couple of interesting features about this law that are worthy of note. First the boundary integral can be broken into two parts, one over the inflow region $\Gamma_{in}(t)$ and the other over the outflow:

$$\Gamma_{out}(t) = \Gamma - \Gamma_{in}(t) .$$

On the latter $\frac{\partial\psi}{\partial\sigma} \geq 0$ (see (2.9)); hence the contribution to (2.15) from the outflow is non-negative, i.e.,

$$(2.16) \qquad \frac{\partial}{\partial t}\left\{ ||\zeta(\cdot, t)||_o^2 \right\} \leq - \int_{\Gamma_{in}(t)} \left(\frac{\partial\psi}{\partial\sigma}\right) \zeta^2 \quad .$$

From this we see that the time growth of the potential entrosphy is bounded by the initial data, which we take as

$$(2.17) \qquad \zeta(\underline{x}, 0) = \zeta_o(\underline{x}) \qquad \underline{x} \in \Omega ,$$

$$(2.18) \qquad \psi(\underline{x}, 0) = \psi_o(\underline{x}) \qquad \underline{x} \in \Omega ,$$

and the boundary data on the inflow. This in essense is the raison d'etre for (2.10).

The equality (2.16) is quite central to the entire problem, and plays a role similar to the coerciveness inequality for elliptic equations. In fact, using the techniques of [16], uniqueness and continuous dependence on data can be readily developed from (2.16), and in Section 3 we shall develop a similar inequality for finite elements and use it to prove stability and convergence.

3. Finite element approximation. Our approximation to the barotropic model (2.5), (2.7)-(2.8), (2.10), (2.17)-(2.18) is obtained by the classical Galerkin (or method of weighted residuals) used in conjunction with finite elements. The first step is to subdivide Ω into triangles (possibly curvilinear at the boundary), and to consider a finite element space on this grid.

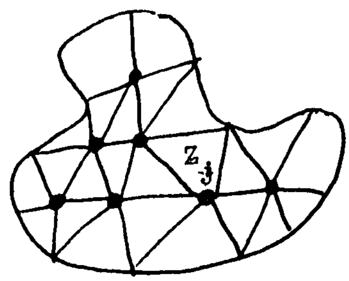

Figure 3.1.- Discretization of Ω.

In practice attention will be restricted to three specific finite elements, although the theory developed in the next section applies much more generally. The simplest element we shall use is the familiar linear piecewise polynomial function. Letting $\underline{z}_j$, $1 \leq j \leq N^h$ (*), be the nodes of the triangles, any function $v^h(\underline{x})$ in the piecewise linear finite element space S^h can be written

$$(3.1) \qquad v^h(\underline{x}) = \sum_{j=1}^{N^h} v^h(\underline{z}_j)\ \phi_j(\underline{x})$$

(*) In the sequel $h > 0$ will denote an average mesh spacing, and for convenience quantities associated with finite elements will be parameterized by h; e.g., the number of nodes is denoted N^h.

that are worthy of note. First the boundary integral can be broken into two parts, one over the inflow region $\Gamma_{in}(t)$ and the other over the outflow:

$$\Gamma_{out}(t) = \Gamma - \Gamma_{in}(t) .$$

On the latter $\frac{\partial\psi}{\partial\sigma} \geq 0$ (see (2.9)); hence the contribution to (2.15) from the outflow is non-negative, i.e.,

$$(2.16) \qquad \frac{\partial}{\partial t}\left\{ ||\zeta(\cdot, t)||_0^2 \right\} \leq -\int_{\Gamma_{in}(t)} \left(\frac{\partial\psi}{\partial\sigma}\right) \zeta^2 \quad .$$

From this we see that the time growth of the potential entrosphy is bounded by the initial data, which we take as

$$(2.17) \qquad \zeta(\underline{x}, 0) = \zeta_0(\underline{x}) \qquad \underline{x} \in \Omega ,$$

$$(2.18) \qquad \psi(\underline{x}, 0) = \psi_0(\underline{x}) \qquad \underline{x} \in \Omega ,$$

and the boundary data on the inflow. This in essense is the raison d'etre for (2.10).

The equality (2.16) is quite central to the entire problem, and plays a role similar to the coerciveness inequality for elliptic equations. In fact, using the techniques of [16], uniqueness and continuous dependence on data can be readily developed from (2.16), and in Section 3 we shall develop a similar inequality for finite elements and use it to prove stability and convergence.

3. Finite element approximation. Our approximation to the barotropic model (2.5), (2.7)-(2.8), (2.10), (2.17)-(2.18) is obtained by the classical Galerkin (or method of weighted residuals) used in conjunction with finite elements. The first step is to subdivide Ω into triangles (possibly curvilinear at the boundary), and to consider a finite element space on this grid.

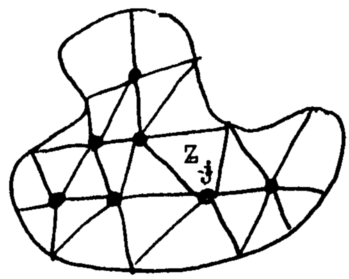

Figure 3.1.- Discretization of Ω.

In practice attention will be restricted to three specific finite elements, although the theory developed in the next section applies much more generally. The simplest element we shall use is the familiar linear piecewise polynomial function. Letting $\underline{z}_j$, $1 \leq j \leq N^h$ (*), be the nodes of the triangles, any function $v^h(\underline{x})$ in the piecewise linear finite element space S^h can be written

$$(3.1) \qquad v^h(\underline{x}) = \sum_{j=1}^{N^h} v^h(\underline{z}_j)\, \phi_j(\underline{x})$$

(*) In the sequel $h > 0$ will denote an average mesh spacing, and for convenience quantities associated with finite elements will be parameterized by h; e.g., the number of nodes is denoted N^h.

where ϕ_j are the familiar hill functions [7] satisfying

$$\phi_j(\underline{z}_\ell) = \begin{cases} 1 \text{ if } j = \ell, \\ 0 \text{ if } j \neq \ell. \end{cases} \tag{3.2}$$

We shall also use quadratic and cubic elements. For these spaces any function v^h can be represented by (3.1) except that $\underline{z}_j$ $(1 \leq j \leq N^h)$ include not only nodes but also midpoints in the case of quadratics, and two equally spaced points on the sides of triangles plus the centroid in the case of cubics (see Figure 3.2).

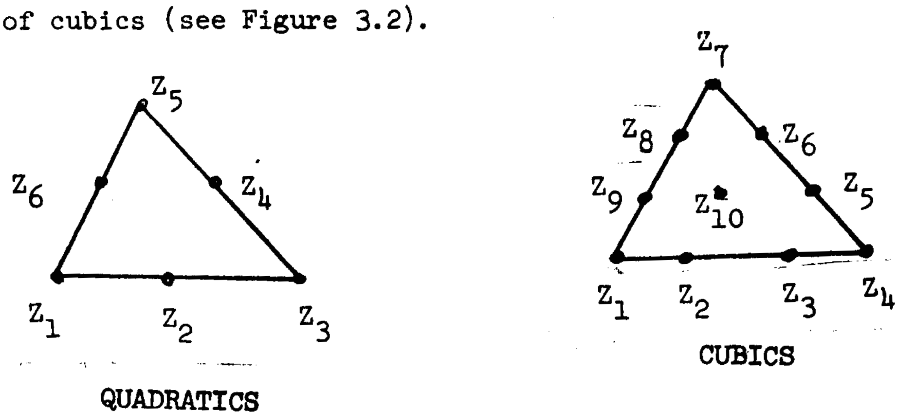

Figure 3.2.- The points z_j.

In each of the above cases we have approximations to the stream function and potential vorticity at each time $t > 0$ of the form

$$\psi^h(\underline{x}, t) = \sum_{j=1}^{N^h} \psi_j^h(t)\ \phi_j(\underline{x}), \tag{3.3}$$

$$\zeta^h(\underline{x}, t) = \sum_{j=1}^{N^h} \zeta_j^h(t)\ \phi_j(x). \tag{3.4}$$

We determine the weights $\psi_j^h(t) = \psi^h(\underline{z}_j, t)$ and $\zeta_j^h(t) = \zeta^h(\underline{z}_j, t)$ by the

Galerkin method and the initial and boundary conditions. For the latter we have from (2.8) and (2.10)

$$\psi_j^h(t) = \psi_\Gamma(\underline{z}_j, t) \qquad \text{for } \underline{z}_j \in \Gamma,\ t > 0\ , \tag{3.5}$$

$$\zeta_j^h(t) = \zeta_\Gamma(\underline{z}_j, t) \qquad \text{for } \underline{z}_j \in \Gamma_{in}(t),\ t > 0\ , \tag{3.6}$$

and for initial conditions we take

$$\psi_j^h(0) = \psi_o(\underline{z}_j)\ , \tag{3.7}$$

$$\zeta_j^h(0) = \zeta_o(\underline{z}_j)\ . \tag{3.8}$$

To determine the remaining weights, namely

$$\psi_j^h(t) \text{ for } \underline{z}_j \in \Omega\ ,\ \zeta_j^h(t) \text{ for } \underline{z}_j \in \Omega \cup \Gamma_{out}(t)\ ,$$

we rewrite the barotropic equations in the integral form

$$\iint_\Omega \frac{\partial\zeta}{\partial t} v = \iint_\Omega J(\zeta, \psi) v \qquad \text{all } v \in L_2(\Omega)\ , \tag{3.9}$$

$$\iint_\Omega \nabla\psi\nabla v = \iint_\Omega (f - \zeta) v \qquad \text{all } v \in H_o^1(\Omega)^{(*)}\ . \tag{3.10}$$

The Galerkin idea is to use (3.9)-(3.10) except restricted to a finite

---

(*) In keeping with standard notation we let $H^1(\Omega)$ be the space of functions v for which $||v||_1 < \infty$, and $H_o^1(\Omega)$ be the subspace of $H^1(\Omega)$ of functions vanishing on Γ [16].

dimensional subspace, which in our setting is the finite element space S^h. As usual we require the test functions v to satisfy the homogeneous boundary conditions, hence the approximate system is equivalent to

$$\iint_{\Omega} \frac{\partial \zeta^h}{\partial t} \phi_j = \iint_{\Omega} J(\zeta^h, \psi^h)\, \phi_j \quad \text{all } \underline{z}_j \in \Omega \cup \Gamma_{out}(t), \tag{3.11}$$

$$\iint_{\Omega} \nabla\psi^h \cdot \nabla\phi_j = \iint_{\Omega} (f - \zeta^h)\, \phi_j \quad \text{all } \underline{z}_j \in \Omega . \tag{3.12}$$

Observe that (3.11) is an implicit system of ordinary differential equations in the weights

$$\underline{\zeta}^h(t) = \{\zeta_j^h(t)\} \quad , \underline{\psi}^h(t) = \{\psi_j^h(t)\} .$$

which can be written

$$M\dot{\underline{\zeta}}^h = \underline{J}^h(\underline{\zeta}^h, \underline{\psi}^h), \ \dot{\underline{\zeta}}^h = \frac{d}{dt} \zeta^h , \tag{3.13}$$

where the "mass matrix" M has the entries

$$\iint_{\Omega} \phi_j\ \phi_k \qquad (\underline{z}_k \in \Omega \cup \Gamma \text{ and } \underline{z}_j \in \Omega \cup \Gamma_{out}(t))$$

and J^h is a vector whose generic entries are

$$\iint_{\Omega} J(\zeta^h, \psi^h)\phi_j \quad (\underline{z}_j \in \Omega \cup \Gamma_{out}(t)) .$$

The elliptic equation (3.12) is equivalent to the matrix equation

(3.14) $$K\underline{\psi}^h = \underline{f} - M\underline{\zeta}^h$$

where K is the "stiffness matrix"

$$\iint_{\Omega} \nabla\phi_k \nabla\phi_j \qquad (\underline{z}_k \in \Omega \cup \Gamma,\ \underline{z}_j \in \Omega),$$

and the entries of the vector $\underline{f}$ are $\iint_{\Omega} f\phi_j$ $(\underline{z}_j \in \Omega)$.

In practice of course it is necessary to solve both the ordinary differential equation (3.13) and the matrix equation (3.14). The latter causes very few problems. Once the dependent variables are removed from ψ^h (i.e., the ones determined by the boundary conditions (3.5)), the resulting coefficient matrix is positive definite. Hence it can be factored (by elimination) into LDL^T where L is triangular and D is diagonal. Once this is done (3.14) can be solved for the stream function in terms of the vorticity at any time $t > 0$ by simple backsolves.

The time integration of (3.13) is a more delicate matter. First, this system is stiff -- the eigenvalues of the associated linearized system vary from $O(1)$ to $O(N^h)^{1/2}$. Secondly, the system is implicit. The mass matrix M is a sparse banded matrix, but it is not diagonal. Moreover, direct inversion of M to relate the time derivatives of $\underline{\zeta}^h$ to values of $\underline{\zeta}^h$ and $\underline{\psi}^h$ must be rejected since M^{-1} is typically a full matrix.

It is perhaps appropriate to digress slightly at this point and mention that the most serious criticism of the finite element ideas has been the rather fundamental role implicitness plays in the latter [12]. Could one not develop difference schemes of comparable accuracy which do not involve mass matrices like M? This of course is an important and serious criticism and one which will be dealt with in the final section containing numerical results.

The implicitness of (3.13) can be avoided by a very simple process known as lumping. The key observation is that

$$\iint_\Omega \phi_j \phi_k = 0$$

unless z_j and z_k belong to a common triangle. In the latter case $\dot{\zeta}_k^h$ is equal to $\dot{\zeta}_j^h$ plus terms of order $h^{(*)}$. Thus (3.13) can be replaced by

$$\left(\sum_{k=1}^{N^h} \iint_\Omega \phi_k \phi_j\right) \dot{\zeta}_j^h = \iint J(\zeta^h, \psi^h)\phi_j$$

(with error O(h)) which in matrix form becomes

$$D\dot{\underline{\zeta}}^h = \underline{J}(\underline{\zeta}^h, \underline{\psi}^h)$$

where D is diagonal. The approximation is then completed by using an explicit forward time difference for $\dot{\underline{\zeta}}^h$ after a premultiplication by D^{-1}.

While this technique is quite popular in certain engineering circles [7], it is not without serious difficulties. First, an error of O(h) is made in the lumping process; hence it cannot be used efficiently for higher order elements such as quadratics and cubics. In addition there are problems with lumping even for linear elements. For example consider

---

(*)For a uniform grid cancellation occurs and $\dot{\zeta}_k^h = \dot{\zeta}_j^h + O(h^2)$.

a one dimensional linearized advection of vorticity

$$\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} = 0 .$$

With linear elements and a uniform mesh the analog of (3.13) is

$$(\dot{\zeta}^h_{j-1} + 4\dot{\zeta}^h_j + \dot{\zeta}^h_{j+1})/6 + U(\zeta^h_{j+1} - \zeta^h_{j-1})/2h = 0 .$$

The associated lumped system is

$$\dot{\zeta}^h_j + U(\zeta^h_{j+1} - \zeta^h_{j-1})/2h = 0 .$$

If one uses an explicit forward time difference this becomes

$$\zeta^h_j([n+1]\Delta t) = \zeta^h_j(n\Delta t) - (U\Delta t/2h)[\zeta^h_{j+1}(n\Delta t) - \zeta^h_{j-1}(n\Delta t)]$$

which is a classical unconditionally unstable difference scheme [15]! It would appear therefore that implicitness is quite fundamental to the finite element approximations, and except for certain quite special circumstances lumping must be rejected for first order hyperbolic equations like the vorticity transport (3.9).

In order to describe some efficient time discretizations for (3.13) let us introduce a time step $\Delta t > 0$ and write

$$\zeta^{(n)}_j = \zeta_j(n\Delta t) , \; \psi^{(n)}_j = \psi_j(n\Delta t) .$$

The most obvious second order accurate scheme that can be used is the leap frog method

$$(3.15) \qquad M(\underline{\zeta}^{(n+1)} - \underline{\zeta}^{(n-1)})/2\Delta t = J(\underline{\zeta}^{(n)}, \underline{\psi}^{(n)})$$

which perhaps is more conveniently written in variational form as

$$(3.15') \qquad \iint_{\Omega} [\frac{\zeta^{(n+1)} - \zeta^{(n-1)}}{2\Delta t}]\phi = \iint_{\Omega} J[\underline{\zeta}^{(n)}, \underline{\psi}^{(n)}]\phi .$$

Observe that (3.15) is implicit, but the scheme is not unconditionally stable. In fact, the usual (intuitive) stability analysis [15] gives

$$(3.16) \qquad \frac{U\Delta t}{h_o} \leq 1$$

as the Courant condition, where h_o is the minimum mesh length and

$$U = \sup_{x\in\Omega} \{|\frac{\partial\psi^h}{\partial x}| + |\frac{\partial\psi^h}{\partial y}|\} .$$

If the inflow region $\Gamma_{in}(t)$ is independent of time t, then the mass matrix will not depend on time.(*) This means that at the start of the computation the (necessarily positive definite) mass matrix can be factored into LDL^T and hence (3.15) can be marched in time by simple backsolves.

Unfortunately, in the M.O.D.E. experiments the inflow will vary with time, often quite drastically. This means that the mass matrix will have to be refactored at each time step. Moreover, the grids typically will not be uniform and the minimum mesh length h_o could be quite small.

(*)The coefficients of M are independent of time t but the size of M is NXN where N = N(t) is the number of nodes $\underline{z}_j$ in Ω and on the outflow $\Gamma_{out}(t)$.

In this case the Courant condition (3.16) requires a smaller time interval Δt, and hence more time steps. Because of the expense of the factorization at each time step this would appear to indicate that (3.15) is not very efficient for such cases. A much more attractive alternative is the implicit scheme

$$(3.16) \qquad \iint_{\Omega} \left[\frac{\zeta^{(n+1)} - \zeta^{(n-1)}}{2\Delta t}\right]\phi = \iint_{\Omega} J\left(\left[\frac{\zeta^{(n+1)} + \zeta^{(n-1)}}{2}, \psi^{(n)}\right]\right)\phi$$

whose linearization is unconditionally stable.

As will be shown in the next section, the Galerkin "semi-discrete" system (3.11)-(3.12) produces approximations to the velocities

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}$$

to order $O(h^{k-1})$ if piecewise polynomials of degree k-1 are used. Similar approximations are made to the vorticity. Thus the second order schemes (3.15)-(3.16) would seem to be appropriate for the linear and quadratic elements but not for cubics where third order spatial approximations are obtained. As will be reported in the last section, however, we found (3.15) or (3.16) quite suitable for cubics provided time extrapolation is used.

4. Stability and convergence. The goal of this section is to prove that the semi-discrete finite element system (3.11)-(3.12) is stable in the sense that

$$||\zeta^h(\cdot, t)||_o, \; ||\psi^h(\cdot, t)||_1$$

are bounded by the initial and boundary data. In addition, the order of convergence of the approximate vorticity ζ^h and stream function ψ^h is established.

The proof of stability centers around a calculation similar to one which produced (2.15). We start by writing

$$\zeta^h = \zeta^h_{\Omega} + \zeta^h_{\Gamma} \, , \; \psi^h = \psi^h_{\Omega} + \psi^h_{\Gamma} \tag{4.1}$$

where

$$\zeta^h_{\Gamma} = \sum_{\underline{z}_j \varepsilon \Gamma_{in}(t)} \zeta^h_j \phi_j \, , \; \psi^h_{\Gamma} = \sum_{\underline{z}_j \varepsilon \Gamma} \psi^h_j \phi_j \, . \tag{4.2}$$

Thus, ζ^h_{Ω} vanishes on the inflow $\Gamma_{in}(t)$ and ψ^h_{Ω} vanishes everywhere on the boundary Γ. Because of the former we can replace ϕ_j with ζ^h_{Ω} in (3.11). This gives

$$\begin{aligned}\iint_{\Omega} \zeta^h_{\Omega} \frac{\partial \zeta^h_{\Omega}}{\partial t} = - \iint_{\Omega} \zeta^h_{\Omega} \frac{\partial \zeta^h_{\Gamma}}{\partial t} \\ + \iint_{\Omega} J(\zeta^h_{\Gamma}, \psi^h)\zeta^h_{\Omega} + \iint_{\Omega} J(\zeta^h_{\Omega}, \psi^h)\zeta^h_{\Omega} \, .\end{aligned} \tag{4.3}$$

The left hand side of (4.3) can be written

$$||\zeta_\Omega^h(\cdot,\ t)||_o \frac{d}{dt} ||\zeta_\Omega^h(\cdot,\ t)||_o \quad ;$$

hence after dividing by $||\zeta_\Omega^h(\cdot,\ t)||_o$, (4.3) becomes a differential relation describing the time growth of vorticity. We want to bound the right hand side by known quantities which in this setting are the boundary data ζ_Γ^h, ψ_Γ^h and the initial data. The first term on the right hand side causes no difficulties, and we use the Schwartz inequality to bound it by

$$||\zeta_\Omega^h(\cdot,\ t)||_o \ ||\frac{\partial\zeta_\Gamma^h}{\partial t}(\cdot,\ t)||_o$$

Another application of the Schwartz inequality permits us to bound the second term by

$$||\zeta_\Omega^h(\cdot,\ t)||_o \ ||\zeta_\Gamma^h(\cdot,\ t)||_{1,\infty} \ ||\psi^h(\cdot,\ t)||_1$$

where

$$||v||_{1,\infty} = \sup_{\underline{x}\in\Omega} (|\nabla v| + |v|)$$

The approximate stream function ψ^h and vorticity ζ^h are related through the elliptic system (3.12). Thus standard estimates, [7], give

$$(4.4) \qquad ||\psi^h||_1 \leq C_1||\psi_\Gamma^h||_1 + C_2||\zeta_\Omega^h||_o + C_3||\zeta_\Gamma^h||_o$$

where C_i, $i = 1,2,3$, are constants independent of $h > 0$.

The last expression in (4.3) is the more interesting one. On the surface it appears to be troublesome since $J(\zeta_\Omega^h,\ \psi^h)$ involves first spatial derivatives of ζ_Ω^h.

It is at this point, however, where the inflow condition (3.6) plays a central role. In particular, we duplicate the calculation starting with (2.14) and leading to (2.16) except in this case with functions in the space S^h. We recall that the term in question is

$$\iint_\Omega J(\zeta_\Omega^h, \psi_\Omega^h)\, \zeta_\Omega^h \ .$$

An integration by parts shows that this term is equal to

$$-\oint_\Gamma \left(\frac{\partial \psi^h}{\partial \sigma}\right)(\zeta_\Omega^h)^2/2 \ .$$

But ζ_Ω^h vanishes on the inflow and the tangential derivative of the stream function (i.e., the normal velocity) is positive on the outflow if h is sufficiently small. Hence

$$\iint_\Omega J(\zeta_\Omega^h, \psi_\Omega^h)\, \zeta_\Omega^h \leq 0 \ .$$

Combining the above with (4.3) gives

$$\text{(4.5)} \qquad \frac{d}{dt} ||\zeta_\Omega^h(\cdot, t)||_o \leq C_A + C_B \, ||\zeta_\Omega^h(\cdot, t)||_o$$

where

$$C_A = \sup_{0 \leq \tau \leq t} \left\{ \left|\left| \frac{\partial \zeta_\Gamma^h}{\partial \tau} (\cdot, \tau) \right|\right|_o + ||\zeta_\Gamma^h(\cdot, \tau)||_{1,\infty} [C_1 ||\psi_\Gamma^h(\cdot, \tau)||_1 + C_3 ||\zeta_\Gamma^h(\cdot, \tau)||_o] \right\}$$

$$C_B = \sup_{0 \leq \tau \leq t} \{ ||\zeta_\Gamma^h(\cdot, \tau)||_{1,\infty} \, C_2 \} \ .$$

An integration of the differential inequality (4.5) gives

(4.6) $$||\zeta^h_{\sim}(\cdot, t)||_o \leq \exp[C_B t]||\zeta^h_{\sim}(\cdot, 0)||_o + (C_A/C_B)\{\exp[C_B t] - 1\} .$$

This expression with (4.4) consists of our statement of stability.

We now consider the order of accuracy of the finite element approximations. In particular suppose S^h consists of piecewise polynomials of degree k-1; thus k=2 for linear elements; k=3 for quadratics and k=4 for cubics. Let

$$\tilde{\zeta}^h(\underline{x}, t) = \sum_j \zeta(\underline{z}_j, t)\phi_j ,$$

$$\tilde{\psi}^h(\underline{x}, t) = \sum_j \psi(\underline{z}_j, t)\phi_j ,$$

be the interpolants of ζ, ψ respectively. Then

$$||\tilde{\zeta}^h - \zeta||_r \leq Ch^{k-r} ||\zeta||_k ,$$

$$||\tilde{\psi}^h - \psi||_r \leq Ch^{k-r} ||\psi||_k ,$$

for r=0, 1 where

$$||v||_k = \left(\sum_{\alpha_1+\alpha_2\leq k} \left|\left| \frac{\partial^{\alpha_1+\alpha_2} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|\right|_o^2 \right)^{1/2} .$$

Since $\tilde{\zeta}^h$ is "close to" ζ it is "almost" a solution of the vorticity transport equation (3.9). More precisely, for any $v \in L_2(\Omega)$

$$E^h(v) = \iint_\Omega \frac{\partial\tilde{\zeta}^h}{\partial t} v - \iint_\Omega J(\tilde{\zeta}^h, \psi)v$$

$$= -\left\{ O(h^k)\ ||\frac{\partial\zeta}{\partial t}||_k + O(h^{k-1})\ ||\zeta||_k \right\} ||v||_o \ .$$

Thus as $e^h = \tilde{\zeta}^h - \zeta^h$ vanishes on $\Gamma_{in}(t)$, (3.11) can be replaced with

$$\iint_\Omega e^h \frac{\partial e^h}{\partial t} = \iint_\Omega \left\{ J(\tilde{\zeta}^h, \psi) - J(\zeta^h, \psi^h) \right\} e^h + E^h(e^h) \ .$$

We rewrite this relation as

$$||e^h||_o \frac{d}{dt} ||e^h||_o = \iint_\Omega J(e^h, \psi^h)e^h + \iint_\Omega J(\tilde{\zeta}^h, \psi - \psi^h)e^h + E^h(e^h) \ .$$

The calculation now resembles the stability estimate with the decisive step being the observation that

$$\iint_\Omega J(e^h, \psi^h)e^h \leq 0 \ .$$

With this we conclude

$$\frac{d}{dt} ||e^h||_o \leq ||\tilde{\zeta}^h||_{1,\infty}\ ||\psi - \psi^h||_1 + O(h^{k-1}) \ .$$

But elliptic theory [7] gives

$$||\psi - \psi^h||_1 = O(h^{k-1}) + ||e^h||_o\ O(1),$$

hence

$$||\zeta^h(\cdot, t) - \tilde{\zeta}^h(\cdot, t)||_o = ||e^h(\cdot, t)||_o = O(h^{k-1})$$

since this is true at t=0. We conclude that

$$||\psi(\cdot, t) - \psi^h(\cdot, t)||_1 = O(h^{k-1}) ,$$

$$||\zeta(\cdot, t) - \zeta^h(\cdot, t)||_o = O(h^{k-1}) .$$

Defining the approximate velocity by

$$u^h = -\frac{\partial\psi^h}{\partial y} , \quad v^h = +\frac{\partial\psi^h}{\partial x} \quad (*) ,$$

we see that the finite element system (3.11)-(3.12) produces $O(h^{k-1})$ approximations to the velocities u, v, and a similar order of approximation to the vorticity ζ .

(*) In practice difference quotients of ψ^h are preferable if uniform meshes are used (see [8]).

5. <u>Numerical results</u> - In this section we shall report some preliminary and extremely elementary numerical experiments. As mentioned in the introduction, the calculations based on the M.O.D.E. data will be done with a three dimension quasi-geostrophic model. This work will be done in conjunction with J. Hirsh and A. R. Robinson, and will be reported elsewhere. Here our basic goal will be to confirm the order of accuracy estimates established in Section 4, and in addition give some simple indication of how the advection of spikes is approximated by finite elements.

Our first experiment treats a neutral Rossby wave [17]

$$\psi = -U_y + A \sin [k (x - ct + \alpha y)] ,$$

in a square region $\Omega = [0,L]x[0,L]$. For this to be a solution of the barotropic equation the phase speed c must satisfy

$$c = U - \beta/k^2 (1 + \alpha^2) .$$

Two finite element spaces, namely quadratics and cubics, are considered along with the first order windward difference scheme [18]. In order to keep the amount of work roughly constant, we used a uniform mesh of $h = L/4$ and $L/8$ for the difference scheme with $h = L/2$ and $h = L/4$ for the finite elements. The results are given in Table I for selected points in the region Ω. The time integration was done for the finite element with the implicit scheme discussed in Section 3. The time steps τ satisfied the Courant condition

$$\tau \leq \frac{(h/L)}{U + kA} \tag{5.1}$$

and one extrapolation was used for the cubics per time step. For the explicit upwind scheme one half of the value of (5.1) was used.

TABLE I

Neutral Rossby wave $U = 5,\ A = 100,\ L = 100,\ k = .075,\ \beta = .00125$				
Percent error in $v = \frac{\partial\psi}{\partial x}$				
POINT		(L/4, L/2)	(L/2, L/2)	(3L/4, L/2)
Upwind scheme	h = 1/4	20%	20%	6%
	h = 1/8	8%	10%	6%
Quadratics	h = 1/2	8%	15%	18%
	h = 1/4	3%	4%	5%
Cubics	h = 1/2	3%	3%	4%
	h = 1/4	0.3%	0.4%	1%
Percent error in vorticity				
POINT		(L/4, L/2)	(L/2, L/2)	(3L/4, L/2)
Upwind scheme	h = 1/4	15%	37%	40%
	h = 1/8	10%	21%	23%
Quadratics	h = 1/2	18%	32%	8%
	h = 1/4	5%	6%	2%
Cubics	h = 1/2	2%	10%	5%
	h = 1/4	0.1%	2%	0.6%

The orders of accuracy predicted in Section 4 can in essence be seen from Table I. The slight variation in the rate is due to the fact that the theory in Section 4 was developed for L_2 convergence while Table I refers to pointwise convergence. Observe also that for this simple Rossby wave the higher order elements (i.e., cubic and quadratic) are strikingly more efficient. The approximations obtained from linear elements are almost identical to those listed for the upwind scheme and hence the former was not included.

In our second experiment we consider a stream function of the form

$$\psi = -Uy + A\ f\ [k\ (x - Ut)]$$

with no β-plane effect (i.e., $\beta = 0$ and a constant Coriolis parameter.) The function f is chosen so that the vorticity

$$\zeta = k^2\ A\ f''\ [k(x - Ut)]$$

is a spike:

$$f''(\xi) = \begin{cases} (\sin \pi\ \xi)^3 \text{ if } 0 < \xi < 1 & , \\ 0 \qquad \text{if elsewhere} & . \end{cases}$$

We use the same approximations as in the first experiment but with different data. In particular, we want the support of ζ to be small compared to L and so we use

$$L = 1,\ k = 4 \quad ,$$

giving us the profile shown in figure 5-1. In addition we use

$$A = 1,\ U = 5,\ \alpha = 1.$$

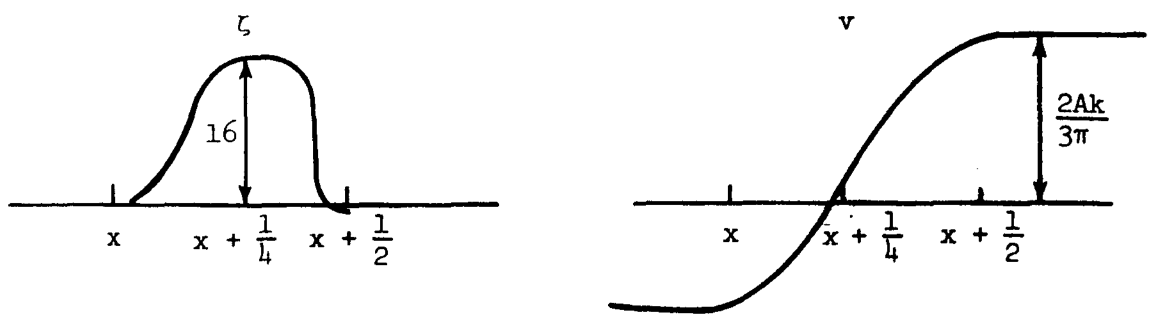

Figure 5-1.- Vorticity spike.

The same time integrations as in the first experiment are used with the time step τ satisfying

(5.2) $$\tau = \frac{h}{U + kA} \quad .$$

To keep the work roughly the same we used one half of the value of τ for the explicit upwind scheme. The profiles for ζ^h and v^h are shown in Figure 5.2.

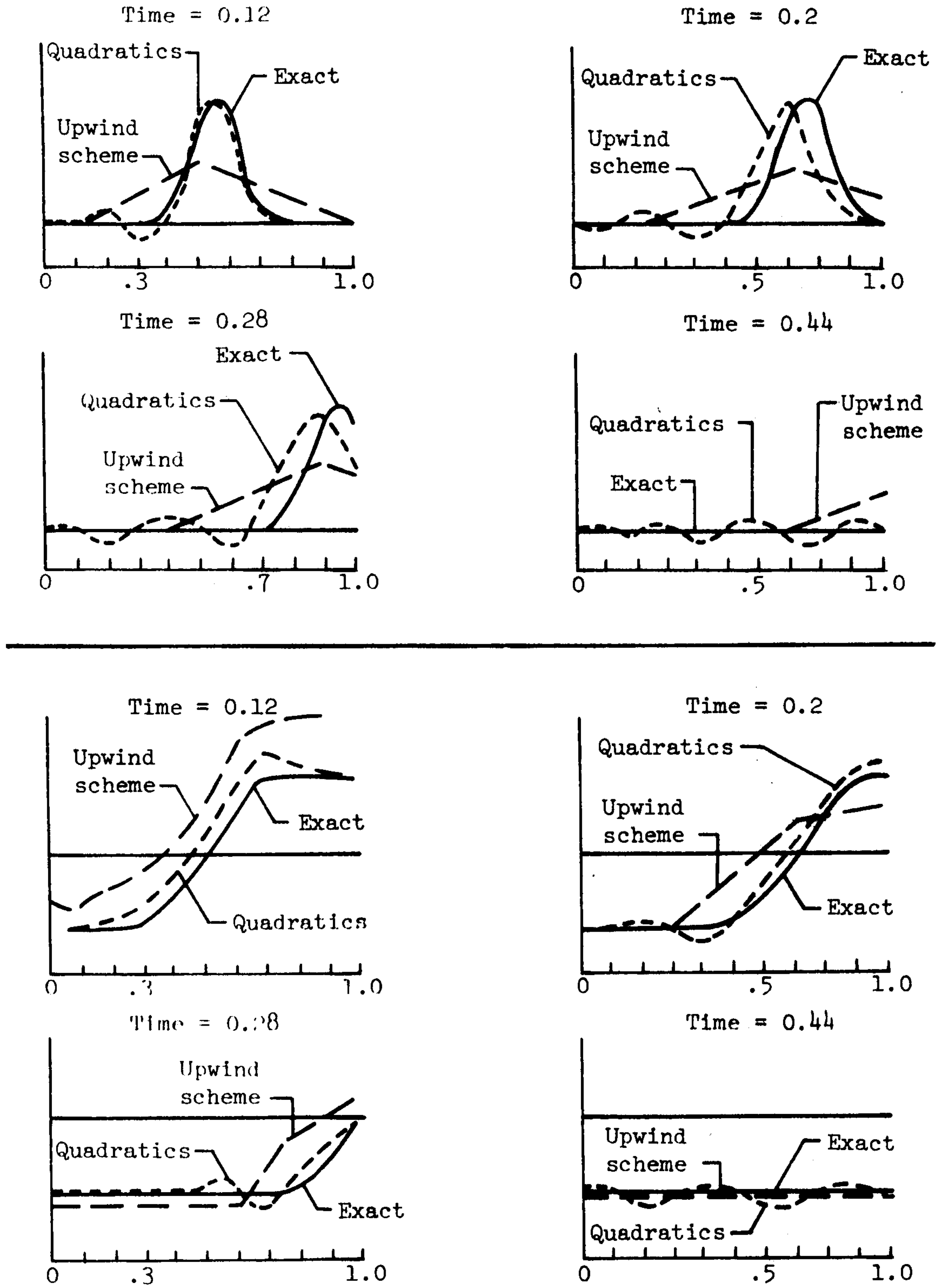

Figure 5.2.- Profiles for ζ^h and v^h $(y = 0.5)$.

It can be observed from Figure 5-2 that the finite element schemes have oscillations downstream from the spike while no such phenomena exists for the upwind difference scheme. This is due to the numerical viscosity in the latter; i.e., upwind differencing is consistent with the dissapative operator

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta = \nu\,\Delta\zeta \tag{5.2}$$

up to terms of order $O(h^2) + \left((\Delta t^2)\right)$. The numerical viscosity coefficient ν is of order $O(h) + O(\Delta t)$, (see [18]). The "numerical friction" dampens oscillations, but also quite unfortunately the spike itself! The "smearing" effect can be readily seen from the vorticity profiles.

The finite element schemes (with the time approximation (3.16)), on the other hand, are conservative in the sense that the numerical viscosity ν is zero. This is easily verified by Taylor expansion, which due to their length will not be produced here.

In most problems, and in particular the MODE calculations, the oscillations (typically with a period of a mesh length) are far less troublesome than smearing. First of all, they are of the right order (e.g., $O(h^2)$ for quadratics), and secondly they are easily detected in a given calculation.

This has interesting implications for the point raised in Section 3 concerning the fundamental role played by implicitness in the finite element method. The fact that the inflow region changes significantly in time--as it does even for the simple examples considered in this Section--eliminates the really attractive higher order, explicit, and conservative difference schemes (e.g., the fourth order scheme of Arakawa [18, p. 105]). If one desires an explicit approximation some form of "windward" differencing at the outflow seems inevitable, and the latter is inherently non-conservative (and inaccurate). In other words, implicitness may be essential for these types of problems, a fact which augers well for the future application of the finite elements ideas in this type of setting.

6. Conclusions. A theory is developed which shows that the finite element method produces stable and accurate approximations to flow problems. While the setting for the latter was the two dimensional stream function-vorticity formulation, the tools of analysis used are rather general, and one can be assured that the method appropriately implemented will produce reliable approximation.

The big issue concerning the use of finite elements centers therefore on efficiency. As the analysis and examples in this paper show, implicitness is quite fundamental to the method. Modifications such as "lumping" which in essence produce explicit approximations can be dangerously unstable in the Navier-Stokes setting, and therefore must be rejected except for very special circumstances. The question is whether the implicitness is really necessary.

For problems with simple geometries (e.g., rectangular polygons) where uniform grids are sufficient, the answer is clearly negative. A rather striking example is obtained by comparing the explicit Arakawa difference scheme with the scheme obtained from cubic finite elements. The simplicity of the former compared to the latter is considerable. They both produce third order approximations to the velocities, yet a very conservative and ad hoc guess would be that the cubics would require at least one and possibly two orders of magnitude more work and computer storage.

This does not mean, however, that finite elements will be useless for flow problems. Indeed, it is now increasingly clear that suitably formulated higher order methods -- either finite differences or finite elements -- can be considerably more efficient than first and second order methods even for complicated engineering problems (see e.g. [7], [11], [12]). This is also clearly illustrated in very simple examples of the previous section. The most striking property of the finite element method is the ease with which higher order approximations can be derived even in the presence of irregular grids and irregular boundaries. This is in striking contrast to explicit higher order finite difference schemes, and for problems such as the limited region ocean model where either irregular grids or boundaries are essential, we feel the finite element ideas can play a significant role.

REFERENCES

1. Robinson, A. R., "Oceanography" in Research Frontiers in Fluid Mechanics, Seeger, R. J., and Emple, R., Ed., Interscience, New York, 1965.

2. Robinson, A. R., "Boundary Layers in Ocean Circulation Models," Ann. Review of Fluid Mechanics, Vol. 2. (1970) 293-312.

3. Greenspan, H. D., "The Theory of Rotating Fluids," Cambridge University Press, 1968.

4. Robinson, A. R., "The Gulf Stream", Phil. Trans. Soc. Lon. 270 (1971) 351-370.

5. Proceedings of Conference on Numerical Models for Ocean Circulation Problems, National Academy of Science, to appear.

6. MODE - I, The Program and the Plan, MODE - I, Scientific Council, The International Decade of Ocean Exploration - NSF, March 1, 1973.

7. Strang, G., and Fix, G. J., "An Analysis of the Finite Element Method," Prentice-Hall, 1973.

8. Fix, G. J., and Nassif, N., "On Finite Element Approximation to Time Dependent Problems", Numer. Math., 19 (1972) 127-135.

9. Baker, A. J., "A Finite-Element Solution Algorithm for Viscous Incompressible Fluid Dynamics," Int. J. Num. Mtd. Engr., 6, (1973) 89-101.

10. Martin, H. C., "Finite Element Analysis of Fluid Flows," Proc. Second Conf. on Matrix Mtds. in Structural Engr., Wright-Patterson AFB, TR-68-150, (1969) 517-535.

11. Argyris, J. H., Mareczek, G., and Scharff, D. W., "Two and Three Dimensional Flows Using Finite Elements," The Aero. Jour. of the Royal Aero. Soc., 73, (1969) 961-964.

12. Kreiss, H. O., Article in reference 5.

13. Pedlosky, J., "The Stability of Currents in the Atmosphere and the Ocean," Jour. of the Atm. Sci., 21 (1964) 201-219.

14. Charney, J. G., Fjortoft, R., von Neumann, J., "Numerical Integration of the Barotropic Vorticity Equation, Tellus, 2, (1950) 237-254.

15. Richtmeyer, R. D., and Morton, K. W., Difference Methods for Initial Value Problems, Interscience, 1967.

16. Lions, J. L., Quélqués Methodes de Résolution de Problems Aux Limités non Lineones, Dunod, Paris, 1970.

17. Platzman, G. W., "The Rossby Wave", Quart. Jour. of the Royal Meterological Soc., 94 (1968) 225-248.

18. Roache, P. J., Computational Fluid Dynamics, Hermosa Pub. Co., Albuquerque, New Mexico.

19. Oden, J. T., and Somogyi, D., "Finite Element Applications in Fluid Dynamics," Jour. of Engr. Mech. Div., ASCE, 95 (1969) 821-826.

20. Oden, J. T., "A Finite Element Analogue of the Navier-Stokes Equations," Jour. of Engr. Mech. Div., ASCE 96 (1970).